Three family unification in higher dimensional models

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Abstract

In orbifold models, gauge, Higgs and the matter fields can be unified in one multiplet from the compactification of higher dimensional supersymmetric gauge theory. We study how three families of chiral fermions can be unified in the gauge multiplet. The bulk gauge interaction includes the Yukawa interactions to generate masses for quarks and leptons after the electroweak symmetry is broken. The bulk Yukawa interaction has global or gauged flavor symmetry originating from the R symmetry or bulk gauge symmetry, and the Yukawa structure is restricted. When the global and gauged flavor symmetries are broken by orbifold compactification, the remaining gauge symmetry which contains the standard model gauge symmetry is restricted. The restrictions from the bulk flavor symmetries can provide explanations of fermion mass hierarchy.

1 Introduction

The standard model (SM) is very well established to describe the physics below the electroweak scale. Theoretically, it is expected that there exists a model beyond SM. The conceptual motivation to consider the model beyond SM is to understand the variety of particles as well as the parameters in SM. In fact, the content of particles in SM is a collection of widely disparate fields: gauge bosons coming in three factors (color, weak and hypercharge), three replicated families of chiral fermions coming in many different representations for quarks and leptons (q, u^c, d^c, ℓ and e^c), and a scalar Higgs boson to break electroweak symmetry and give masses to the chiral fermions. This brings lots of parameters in SM: three gauge couplings, masses and mixings for the quarks and leptons, and a Higgs mass and a Higgs coupling. The Higgs scalar has a quadratic divergence in its mass squared, and thus the electroweak scale is not stable quantum mechanically if the cutoff scale is very high such as the Planck scale. Therefore, there must exist a theory beyond SM around the TeV scale. Besides, the masses and mixings of the quarks and leptons originate from the Yukawa couplings with the Higgs boson, which are the most of the parameters in SM. In such a sense, the nature of the Higgs boson is a key ingredient to go beyond the standard model. It is expected that the Higgs boson will be found at the Large Hadron Collider (LHC) experiment.

The idea of extra dimensions is an attractive candidate to build a model beyond SM. Kaluza and Klein (KK) showed that it is possible to interpret electromagnetism as the effect of gravity in five dimensions under certain projections. After dimensional reduction, a vector/tensor field in some higher dimensional compactified space decomposes into separate scalar, vector and tensor components in four dimensions (4D). The left- and right-handed Weyl fermions in 4D are unified in higher dimensional fermions. The idea of extra dimensions has not been treated in phenomenological issues, but it becomes fashionable since it may explain the large scale hierarchy [1]. The idea of compactification is also applied to break symmetries by orbifold boundary conditions [2, 3, 4]. Though the gauge symmetry is explicitly broken by the orbifold conditions, the gauge couplings can be unified when the brane localized gauge interaction is suppressed by a large volume of the extra dimensions [3]. In such fashions, the idea of gauge-Higgs unification [5] is revived [6]. The scalar Higgs fields can be unified with the gauge fields in some higher dimensional vector fields. In a simple orbifold boundary condition, the gauge symmetry is broken since the broken generators of gauge bosons for 4D coordinates are projected out. At that time, the broken generators for extra dimensional components can have massless modes, and the Wilson line operator can be identified as the Higgs bosons to break the symmetry remained in 4D [7]. Interestingly, this idea is compatible to extend the SM gauge symmetry to a larger gauge group such as in grand unified theories (GUT). Besides, the mass

of the Higgs scalar is forbidden by gauge invariance, and thus it can remain light at low energy when supersymmetry (SUSY) is combined in the model.

The interesting consequence of the gauge-Higgs unification is that the Yukawa interaction can originate from the gauge interaction when fermions are also higher dimensional bulk fields [8, 9]. Actually, the 4D zero modes of fermions can be chiral in orbifold projections, and the higher dimensional extension of the fermion kinetic term with covariant derivative, $\bar{\psi}\gamma^{\mu}(\partial_{\mu} - igA_{\mu})\psi$, includes Yukawa coupling when the gauge fields with higher dimensional components are identified as Higgs fields. In the left-right symmetric construction of the model [10], the matter representation to realize the gauge-Yukawa unification can be much simpler than that of the SM construction, and the actual unification of gauge and Yukawa coupling constants can be realized [9]. In the models of 5D $\mathcal{N} = 1$ SUSY $S^1/\mathbb{Z}_2$ orbifold with bulk gauge symmetries such as $SO(11)$ and $SU(8)$, which break down to Pati-Salam (PS) symmetry group $G_{\mathrm{PS}} = SU(4)_c \times SU(2)_L \times SU(2)_R$ [11] in 4D, matter fields are unified in hypermultiplets, and all three gauge couplings and third generation Yukawa couplings (top, bottom, tau and Dirac tau neutrino) can be unified. Then, the Yukawa couplings at the weak scale can be calculated assuming that the threshold corrections are small. Consequently, we can predict the top quark mass as well as $\tan\beta$, which is a ratio of Higgs vacuum expectation values (VEVs) for up- and down-type Higgs bosons. Actually, the prediction of the top quark mass can agree with the experiment if we take into account the threshold corrections [12]. If we say it inversely, we can survey if the unification of gauge and Yukawa couplings is really realized in the future since the LHC and ILC experiments will provide us more accurate measurement of the Yukawa couplings above TeV scale. It is important that the unification of the gauge and Yukawa coupling constants can be a signal of extra dimensions at ultra high energy scale. Therefore, we should investigate models in which gauge and Yukawa unification can happen.

In SUSY extensions, the matter fermions can be unified in higher dimensional gauge multiplets [13, 14], especially when the model consists of $N = 4$ vector multiplet in 4D language such as in 6D $\mathcal{N} = (1,1)$ SUSY. Interestingly, three replications of family can be obtained in the $T^2/\mathbb{Z}_3$ orbifold [13]. The three families originate from the three chiral supermultiplets in the $N = 4$ gauge multiplet. Since the number of chiral multiplet is maximally three in 4D, it may explain why the family is three times replicated.

The hypermultiplet which is adjoint representation under the bulk gauge symmetry in 5D $\mathcal{N} = 1$ SUSY $S^1/\mathbb{Z}_2$ orbifold model can be incorporated into the gauge multiplet in 6D $\mathcal{N} = (1,1)$ SUSY orbifold models. In Ref.[15], it is found that all matter species for one family and Higgs doublets as well as gauge fields in SM can be unified in 6D $\mathcal{N} = (1,1)$ SUSY $SU(8)$ gauge multiplets with $T^2/\mathbb{Z}_6$ orbifold. The three gauge couplings and the third generation

Yukawa couplings can be also unified in the model. The other gauge groups [16, 17] and other extensions including seven dimensional models [18, 19] are also considered. Since no other bulk matter fields can be introduced, the model can explain why only third family is heavy. If we choose $T^2/\mathbb{Z}_3$ orbifold in the $SU(8)$ model, the discrete charge assignment is simple and almost unique when $N = 1$ SUSY remains at 4D. In the 6D $SU(8)$ $T^2/\mathbb{Z}_3$ orbifold model, thus, three families and Higgs fields as well as gauge fields are naturally unified in one multiplet [20].

In this paper, we will study the cases where there are just three chiral families as zero modes of the bulk fields in the higher dimensional orbifold models. We investigate the Pati-Salam branch of the E_8 and its subgroup to break the symmetry by orbifold compactification, and to extract the three chiral families as zero modes. In fact, the adjoint representation of E_8 contains maximally four families of matters as decomposed representation under the Pati-Salam symmetry. There are three chiral superfields in the bulk for the higher dimensional models, and thus, maximally twelve families are contained in the bulk. By orbifolding, many of the component representations are projected out, and we choose the orbifold charge assignment to extract just three chiral families as zero modes. We will show that there are five cases for the charge assignments to obtain the three chiral families, and give examples of the discrete charge assignments. The Yukawa interaction is generated from the bulk gauge interaction, and we will classify the structure of the mass eigenvalues in the three-family models. In some cases, global or gauged flavor symmetry remains in the Yukawa interaction due to the bulk gauge interaction. The flavor symmetries originates from the R symmetry or gauge symmetry in the bulk. In such cases, two of the eigenvalues are degenerate. In one case, the flavor symmetry is broken by the orbifold projection, and the eigenvalues are not degenerate, but there is a restriction in the 4D gauge symmetry which contains Pati-Salam branch. We study the trinification branch in the typical case to obtain a hierarchical structure of the fermion masses. We also consider the case in which SM gauge symmetry $G_{\mathrm{SM}} = SU(3)_c \times SU(2)_L \times U(1)_Y$ remains in 4D.

This paper is organized as follows: In section 2, we introduce the orbifold models which we consider in this paper. In section 3, we investigate the Pati-Salam branch from the E_8 group and its subgroups. The $\mathbb{Z}_n$ charges are assigned to the decomposed fields under the Pati-Salam symmetry in order break the bulk gauge symmetry down to the Pati-Salam symmetry. In section 4, we will show the five cases to obtain the three chiral families as zero modes from the bulk fields, and give examples of the orbifold charge assignments. In section 5, we make a comment to develop the example to build a model. In section 6, we consider the trinification branch, as well as the SM decomposition. Section 7 is devoted to the conclusions.

2 Gauge, Higgs and matter unification

In this section, we will briefly study the higher dimensional orbifold model, which is used to construct a model where gauge, Higgs and three families of matter are unified in the higher dimensional SUSY gauge multiplet. We will consider 10D $\mathcal{N} = 1$ SUSY model to describe the theory generally, but 8D model can be also considered. Actually, the model examples which we will see later can be made in 8D orbifold. We consider the extra dimensions are compactified over a flat $T^2/\mathbb{Z}_{n_1} \times T^2/\mathbb{Z}_{n_2} \times T^2/\mathbb{Z}_{n_3}$ orbifold. The formalism of the higher dimensional models can be seen in Ref.[21].

From a 4D point of view, the 10D $\mathcal{N} = 1$ gauge multiplet is recognized as one $N = 4$ multiplet which consists of one $N = 1$ vector superfield V and three chiral superfields Σ_i ($i = 1, 2, 3$). The scalar components of the chiral superfields Σ_1, Σ_2, and Σ_3 are $A_5 - iA_6$, $A_7 - iA_8$, and $A_9 - iA_{10}$, respectively. We define the extra dimensional coordinates as $z_1 = x_5 + ix_6$, $z_2 = x_7 + ix_8$, and $z_3 = x_9 + ix_{10}$. The orbifold transformations $\mathbf{R}_i$ are $z_i \to \omega z_i$, where $\omega = e^{2\pi i/n_i}$. The transformation $\mathbf{R}_i$ can also act on the internal symmetry of the Lagrangian. The internal symmetry in our class of models is the product of R symmetry and $Aut(G)$. This extension of $\mathbf{R}_i$ can break SUSY as well as the bulk gauge group G. Depending on the discrete charge assignment, the 4D $N = 4$ SUSY can be broken down to $N = 0, 1, 2$.

If at least $N = 1$ SUSY remains at 4D fixed points, the orbifold conditions of the superfields V and Σ_i are given as

$$
\begin{align}
V(x^\mu, \bar{\omega}\bar{z}_i, \omega z_i) &= R_i[V(x^\mu, \bar{z}_i, z_i)], \tag{1}\\
\Sigma_1(x^\mu, \bar{\omega}\bar{z}_i, \omega z_i) &= \bar{\omega}^{k_i} R_i[\Sigma_1(x^\mu, \bar{z}_i, z_i)], \tag{2}\\
\Sigma_2(x^\mu, \bar{\omega}\bar{z}_i, \omega z_i) &= \bar{\omega}^{l_i} R_i[\Sigma_2(x^\mu, \bar{z}_i, z_i)], \tag{3}\\
\Sigma_3(x^\mu, \bar{\omega}\bar{z}_i, \omega z_i) &= \bar{\omega}^{m_i} R_i[\Sigma_3(x^\mu, \bar{z}_i, z_i)], \tag{4}
\end{align}
$$

where R_i acts on the gauge algebra. Since there is a higher dimensional version of trilinear gauge interaction term in Lagrangian, $k_i + l_i + m_i \equiv 0 \pmod{n_i}$ needs to be satisfied. Also, we need $k_1 = 1$, $l_2 = 1$ and $m_3 = 1$ to make the lagrangian invariant. Therefore, one of (k_i, l_i, m_i) has to be 1. From a geometrical consequence, n_i has to be 2,3,4,6. Possible combinations of (k_i, l_i, m_i) are then the followings up to permutation:

$$
\begin{align}
n_i = 2, \quad & (k_i, l_i, m_i) = (0, 1, 1), \tag{5}\\
n_i = 3, \quad & (k_i, l_i, m_i) = (0, 1, 2), (1, 1, 1), \tag{6}\\
n_i = 4, \quad & (k_i, l_i, m_i) = (0, 1, 3), (1, 1, 2), \tag{7}\\
n_i = 6, \quad & (k_i, l_i, m_i) = (0, 1, 5), (1, 1, 4), (1, 2, 3). \tag{8}
\end{align}
$$

For example, let us consider the case where the orbifolds for z_2, z_3 are trivial (we may call the case $n_2, n_3 = 1$.). If this is the case to build a model, we may consider 6D model instead of 10D model. In this case, $(k_1, l_1, m_1) = (1, l, m)$, and $1 + l + m \equiv 0 \pmod{n_1}$. If $l = 0$, $N = 2$ SUSY remains on the orbifold fixed points. In general, if one of (k_i, l_i, m_i) is zero, $N = 2$ remains on the fixed points from the action $\mathbf{R}_i$. However, if the other torus orbifold condition is non-trivial, the SUSY can be broken down to $N = 1$ for the 4D zero modes. In the examples we will see later, one can break down SUSY down to $N = 1$ as well as the bulk gauge group even if $n_3 = 1$. Therefore, the models can be constructed even in 8D. In that case, the scalar component of Σ_3 is not an extra dimensional gauge bosons.

The scalar components of the superfields Σ_i are higher dimensional gauge fields. As a consequence, the bulk gauge interaction includes the term $f_{abc} \Sigma_1^a \Sigma_2^b \Sigma_3^c$ in the superpotential in 4D, where f_{abc} is a structure constant of the gauge group. If the matter and Higgs fields originate from the gauge multiplet, the bulk gauge interaction can contain the Yukawa coupling to generate the fermion masses by Higgs mechanism. Actually, when we use the conventional normalization of the gauge coupling, the bulk Yukawa coupling constant is the same as the gauge coupling g in 4D. Therefore, it is interesting to consider the possibility that the fermions (quarks and leptons), the gauge bosons and the Higgs fields, all originate as the zero modes from the same higher dimensional gauge multiplet.

3   Charge assignments in Pati-Salam basis

In this section, we will assign the discrete charges to the decomposed representations under Pati-Salam symmetry, which are contained in the adjoint representation of E_8, E_7, and $SO(16)$. Under the Pati-Salam branch, it is clear to see all the chiral matter species in the adjoint representations of bulk gauge symmetries.

It is well known that E_8 has a maximal subgroup $SO(16)$, and thus it can contain $SO(10)$ gauge symmetry with flavor symmetry $SO(6) \simeq SU(4)$. The adjoint representation $\mathbf{248}$ is decomposed under $SO(10) \times SU(4)$ as

$$\mathbf{248} = (\mathbf{45}, \mathbf{1}) + (\mathbf{16}, \mathbf{4}) + (\overline{\mathbf{16}}, \overline{\mathbf{4}}) + (\mathbf{10}, \mathbf{6}) + (\mathbf{1}, \mathbf{15}). \tag{9}$$

The $SO(10)$ has a Pati-Salam subgroup $G_{\mathrm{PS}} = SU(4) \times SU(2) \times SU(2)$, and it is useful to describe the $SO(10)$ algebra in the Pati-Salam basis. Here, we assign the $\mathbb{Z}_n$ charges for the decomposed fields under the Pati-Salam symmetry to break the bulk gauge symmetry E_8 down to the Pati-Salam symmetry.

The $SO(10)$ adjoint field contains the following decomposed representations and the discrete

charges are assigned as follows:

$$
\begin{array}{|c|c|c|c|}
\hline
(\mathbf{15}, \mathbf{1}, \mathbf{1}) & (\mathbf{1}, \mathbf{3}, \mathbf{1}) & (\mathbf{1}, \mathbf{1}, \mathbf{3}) & (\mathbf{6}, \mathbf{2}, \mathbf{2}) \\
\hline
0 & 0 & 0 & e \\
\hline
\end{array}
\tag{10}
$$

The spinor representation $\mathbf{16}$ contains four flavors of matter $L_i : (\mathbf{4}, \mathbf{2}, \mathbf{1})$, and $\bar{R}_i : (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$.

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
L_1 & L_2 & L_3 & L_4 & \bar{R}_1 & \bar{R}_2 & \bar{R}_3 & \bar{R}_4 \\
\hline
x & y & z & w & e+x & e+y & e+z & e+w \\
\hline
\end{array}
\tag{11}
$$

The spinor representation $\overline{\mathbf{16}}$ contains four flavors of anti-matter $\bar{L}^i : (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1})$, and $R^i : (\mathbf{4}, \mathbf{1}, \mathbf{2})$.

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bar{L}^1 & \bar{L}^2 & \bar{L}^3 & \bar{L}^4 & R^1 & R^2 & R^3 & R^4 \\
\hline
-x & -y & -z & -w & e-x & e-y & e-z & e-w \\
\hline
\end{array}
\tag{12}
$$

The vector representation $\mathbf{10}$ includes $SU(4)$ sextet $C^{ij} : (\mathbf{6}, \mathbf{1}, \mathbf{1})$, and bidoublet representation $H^{ij} : (\mathbf{1}, \mathbf{2}, \mathbf{2})$. They are sextet under the flavor symmetry, and the flavor indices ij are anti-symmetric.

$$
\begin{array}{|c|c|}
\hline
C^{ij} & H^{ij} \\
\hline
\begin{pmatrix}
- & z+w & y+w & y+z \\
 & - & x+w & x+z \\
 & & - & x+y \\
 & & & -
\end{pmatrix}
&
\begin{pmatrix}
- & e+z+w & e+y+w & e+y+z \\
 & - & e+x+w & e+x+z \\
 & & - & e+x+y \\
 & & & -
\end{pmatrix} \\
\hline
\end{array}
\tag{13}
$$

Because $\epsilon_{ijkl} C^{ij} C^{kl}$ is a singlet, C^{34} component is a conjugate of C^{12}, for example.

The $SO(10)$ singlet $S_i^{\;j}$ is an adjoint under the flavor symmetry.

$$
\begin{array}{|c|}
\hline
S_i^{\;j} \\
\hline
\begin{pmatrix}
0 & x-y & x-z & x-w \\
-x+y & 0 & y-z & y-w \\
-x+z & -y+z & 0 & z-w \\
-x+w & -y+w & -z+w & 0
\end{pmatrix} \\
\hline
\end{array}
\tag{14}
$$

The bulk gauge interaction includes the Yukawa term $L_i \bar{R}_j H^{ij}$.

There are constraints for the discrete charges x, y, z, w and e due to the algebra:

$$
x + y + z + w \equiv 0, \qquad 2e \equiv 0 \pmod{n}. \tag{15}
$$

When $e \equiv 0$, at least $SO(10)$ symmetry remains because the self-conjugate representation $(\mathbf{6}, \mathbf{2}, \mathbf{2})$ has a zero mode in vector multiplet. Under these conditions, one can find that $T^2/\mathbb{Z}_3$ orbifold can not break $SO(10)$ symmetry for these Pati-Salam decomposition.

As is given, one E_8 adjoint includes 4 flavors in $SO(10)_{\text{GUT}} \times SU(4)_F$ branch. One can also interpret the $SU(4)$ symmetry as $SU(4)_c$, which includes color $SU(3)_c$ symmetry. In the former case, there is a self-conjugate representation $(\mathbf{6}, \mathbf{2}, \mathbf{2})$ under the Pati-Salam decomposition, and thus with $T^2/\mathbb{Z}_3$ orbifold, it is impossible to break $SO(10)$ symmetry as we have noted. In the latter branch of $SU(4)_c \times SO(10)_w$, however, the self-conjugate representation is not included under Pati-Salam symmetry and thus $T^2/\mathbb{Z}_3$ orbifold can break $SO(10)_w$ down to $SU(2)_L \times SU(2)_R \times U(1)^3$. Actually, this is related to the fact that E_8 has two different $SU(8) \times U(1)$ subgroups. One is obtained from E_7 branch, $E_8 \to E_7 \times U(1) \to SU(8) \times U(1)$:

$$\mathbf{248} = \mathbf{63}_0 + \mathbf{28}_2 + \overline{\mathbf{28}}_{-2} + \mathbf{1}_0 + \mathbf{1}_{-4} + \mathbf{28}_{-2} + \mathbf{70}_0 + \overline{\mathbf{28}}_2 + \mathbf{1}_4. \tag{16}$$

The other is obtained from $SU(9)$ branch $E_8 \to SU(9) \to SU(8) \times U(1)$:

$$\mathbf{248} = \mathbf{63}_0 + \mathbf{28}_2 + \overline{\mathbf{28}}_{-2} + \mathbf{1}_0 + \mathbf{8}_{-3} + \mathbf{56}_{-1} + \overline{\mathbf{56}}_1 + \overline{\mathbf{8}}_3. \tag{17}$$

The former one includes a self-conjugate representation $\mathbf{70}$, four-rank anti-symmetric tensor in $SU(8)$, but the latter one does not have any self conjugate representations. The $SU(8)$ adjoint $\mathbf{63}$ includes one flavor of matter under the Pati-Salam decomposition, and $\mathbf{28}$ and $\overline{\mathbf{28}}$ include one more flavor. Therefore, in the latter branch, there are two flavors under the Pati-Salam branch in the adjoint representation. If we use the $SU(9)$ branch, it is easy to obtain the charge assignment for the Pati-Salam decomposed fields. The adjoint representation is decomposed as $\mathbf{248} = \mathbf{80} + \mathbf{84} + \overline{\mathbf{84}}$ under $SU(9)$, where $\mathbf{80}$ is an adjoint under $SU(9)$ and $\mathbf{84}$ is a three-anti-symmetric tensor. When we assign the $\mathbb{Z}_n$ charge as

$$
\begin{array}{|c|c|c|}
\hline
\mathbf{80} & \mathbf{84} & \overline{\mathbf{84}} \\
\hline
0 & a & -a \\
\hline
\end{array}
\tag{18}
$$

E_8 is broken down to $SU(9)$. Under the $SU(9)$ space, acting the unitary rotation matrix

$$R = \text{diag}\left(1, 1, 1, 1, \omega^b, \omega^b, \omega^c, \omega^c, \omega^d\right), \tag{19}$$

we obtain the charge assignments for the component representations in the Pati-Salam branch. For example, the left-handed matter components $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ are in the $\mathbf{80}$ and $\mathbf{84}$, and their charges are found to be $-b$ and $a + b + d$.

We can also obtain $\mathbb{Z}_n$ charges under the Pati-Salam decomposition using the breaking chain $E_7 \to SO(12) \times SU(2) \to SO(10) \times SU(2)_F \times U(1)$. In this case, there are two flavors in the E_7 adjoint representation:

$$\mathbf{133} = (\mathbf{45}, \mathbf{1})_0 + (\mathbf{10}, \mathbf{1})_{-2} + (\mathbf{10}, \mathbf{1})_2 + (\mathbf{1}, \mathbf{1})_0 + (\mathbf{16}, \mathbf{2})_1 + (\overline{\mathbf{16}}, \mathbf{2})_{-1} + (\mathbf{1}, \mathbf{3})_0. \tag{20}$$

The $SO(10)$ adjoint field contains the fields as follows:

$(\mathbf{15}, \mathbf{1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{3}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1}, \mathbf{3})$	$(\mathbf{6}, \mathbf{2}, \mathbf{2})$
0	0	0	e

$$(21)$$

The spinor representation $\mathbf{16}$ contains four flavors of matter $L_i : (\mathbf{4}, \mathbf{2}, \mathbf{1})$, and $\bar{R}_i : (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$, and the spinor representation $\overline{\mathbf{16}}$ contains four flavors of anti-matter $\bar{L}^i : (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1})$, and $R^i : (\mathbf{4}, \mathbf{1}, \mathbf{2})$.

L_1	L_2	$\bar{R}_1$	$\bar{R}_2$	$\bar{L}^1$	$\bar{L}^2$	R^1	R^2
x	y	$e + x$	$e + y$	$-x$	$-y$	$e - x$	$e - y$

$$(22)$$

The vector representation $\mathbf{10}$ includes $SU(4)$ sextet C and $\bar{C} : (\mathbf{6}, \mathbf{1}, \mathbf{1})$, and bidoublet representation H and $\bar{H} : (\mathbf{1}, \mathbf{2}, \mathbf{2})$. The fields C and $\bar{C}$ have opposite $U(1)$ charges.

C	$\bar{C}$	H	$\bar{H}$
$-x - y$	$x + y$	$e - x - y$	$e + x + y$

$$(23)$$

The $SO(10)$ singlets $S_i{}^j$ are the adjoint under the flavor symmetry $SU(2)$.

$S_i{}^j$
$\begin{pmatrix} 0 & x - y \\ -x + y & 0 \end{pmatrix}$

$$(24)$$

Similarly to the previous case, we need $2e \equiv 0$. The bulk interaction includes a term $\epsilon_{ij} L_i \bar{R}_j H$.

We also note that the adjoint representation of $SO(16)$ contains two families of matters in the Pati-Salam basis using the chain $SO(16) \rightarrow SU(8) \times U(1)$ [16]. The adjoint representation of $SO(16)$ is decomposed as $\mathbf{120} = \mathbf{63} + \mathbf{28} + \overline{\mathbf{28}} + \mathbf{1}$ under $SU(8) \times U(1)$, where $\mathbf{63}$ is a adjoint under SU(8) and $\mathbf{28}$ is an anti-symmetric tensor. The charge assignment to obtain the Pati-Salam branch is similar to the case of the $SU(9)$ branch in E_8 we have seen. When we assign the $\mathbb{Z}_n$ charge as

$\mathbf{63}$	$\mathbf{28}$	$\overline{\mathbf{28}}$	$\mathbf{1}$
0	a	$-a$	0

$$(25)$$

$SO(16)$ is broken down to $SU(8) \times U(1)$. Under the $SU(8)$ space, acting the rotation matrix

$$R = \text{diag}\,(1, 1, 1, 1, \omega^b, \omega^b, \omega^c, \omega^c), \tag{26}$$

we obtain the charge assignments for the component representations in the Pati-Salam branch. For example, the left-handed matter components $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ are in the $\mathbf{63}$ and $\mathbf{28}$, and their charges are $-b$ and $a + b$.

4 Examples for three families in the bulk

We will classify the cases where there are just three chiral flavors, which are obtained from the zero modes of the bulk fields. For the Pati-Salam decomposition, E_8 adjoint representation contain four vector-like matter representations, as it is written in the previous section. Since there are three adjoint superfields, Σ_i, there are totally 12 vector-like matters in the bulk. By orbifold projection, we will extract only three flavors of chiral matter.

One can find that there are the following five ways to extract the three flavors. The left-handed matter components $L_i : (\mathbf{4}, \mathbf{2}, \mathbf{1})$ under Pati-Salam symmetry have flavor index i for the $SU(4)$ fundamental representation. In the following, we do not care about the permutation of the flavor indices, as well as the permutation of Σ_1, Σ_2, and Σ_3.

1. The three chiral superfields Σ_1, Σ_2, and Σ_3 include L_1, L_2, and L_3, respectively.

2. The three chiral superfields Σ_1, Σ_2, and Σ_3 include L_1, L_1, and L_2, respectively.

3. The three chiral superfields Σ_1, Σ_2, and Σ_3 include L_1, L_1, and L_1, respectively.

4. Flavor $SU(2)$ symmetry remains in 4D, and Σ_1 and Σ_2 include (L_1, L_2) and L_3 respectively.

5. Flavor $SU(3)$ symmetry remains in 4D, and Σ_1 includes (L_1, L_2, L_3).

In the case 2, we need only two flavors of matter in the adjoint representation, and thus, the bulk symmetry can be E_7 or $SO(16)$. In the case 3, only one flavor of matter is needed, and so, the bulk symmetry can be $SU(8)$ [20]. Also, in the case 3, all three chiral superfields Σ_i must have the same $\mathbb{Z}_{n_i}$ charge assignment $k_i = l_i = m_i$. Therefore, n_i has to be 3 in this case.

In the following, we will give examples of charge assignment in each case.

4.1 Case 1

The three chiral superfields Σ_1, Σ_2, and Σ_3 include L_1, L_2, and L_3 components, respectively. To do this, it is necessary that

$$- k_i + x = 0, \quad -l_i + y = 0, \quad -m_i + z = 0. \tag{27}$$

The equivalence symbol with mod n is omitted in the following. Since $k_i + l_i + m_i = 0$ and $x + y + z + w = 0$, we need $w = 0$. As a consequence, the corresponding component in the vector field must have a zero mode, and thus the 4D gauge symmetry is always higher than

Pati-Salam symmetry in this case. If only $L_4, \bar{L}_4$ components remain massless in the vector multiplet, the 4D gauge symmetry will be $SU(6) \times SU(2) \times U(1)^2$. When R_4 and $\bar{R}_4$ remain mass less in addition to them $(w = e = 0)$, $E_6 \times U(1)^2$ symmetry remains in 4D.

Example 1: Consider the orbifold $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$ $(n_1 = 6, n_2 = 3)$.
The charge assignments are

$$(x, y, z, w)_{Z_6} = (1, 2, 3, 0), \quad (x, y, z, w)_{Z_3} = (1, 1, 1, 0), \quad e_{Z_6} = e_{Z_3} = 0, \tag{28}$$

$$(k_1, l_1, m_1) = (1, 2, 3), \quad (k_2, l_2, m_2) = (1, 1, 1). \tag{29}$$

Under these assignments, one can find that the 4D gauge symmetry is $E_6 \times U(1)^2$. Three chiral fields $\mathbf{27}_i$ are zero modes. Since E_6 symmetry remains in this example, it is better to see the chain $E_8 \to E_6 \times SU(3)$ and the decomposition of the adjoint field:

$$\mathbf{248} = (\mathbf{78}, \mathbf{1}) + (\mathbf{27}, \mathbf{3}) + (\overline{\mathbf{27}}, \overline{\mathbf{3}}) + (\mathbf{1}, \mathbf{8}). \tag{30}$$

In fact, under the above charge assignments, the decomposed fields are rearranged as in the E_6 representation, and the $SU(3)$ symmetry is broken down to $U(1)^2$. The bulk interaction includes the Yukawa term

$$\mathbf{27}_1 \cdot \mathbf{27}_2 \cdot \mathbf{27}_3. \tag{31}$$

When the bidoublet components H^{12}, H^{23}, H^{31} (which are also unified in $\mathbf{27}_i$) get VEVs, the fermions acquire masses. Since $\overline{\mathbf{27}}$ from the product of $\mathbf{27} \times \mathbf{27}$ is symmetric under E_6 algebra, the mass matrix of the fermion is symmetric. This is because $\mathbf{27}$ includes both left- and right-handed matters. Note that the indices i of $\mathbf{27}_i$ are the flavor indices, and the bulk interaction is given as tr $[\Sigma_1, \Sigma_2]\Sigma_3$. As a result, the above Yukawa term is included in the bulk interaction, even though the flavor indices are totally anti-symmetric.

Since the diagonal entries are all zero in the symmetric mass matrix, the eigenvalues of the mass matrix cannot be hierarchical (two of the eigenvalues are nearly degenerate, or all three are of the same order). Therefore, to obtain the realistic fermion masses, we need an assist from the brane-localized terms.

Example 2: Next, let us consider the case where the 4D gauge symmetry is $SU(6) \times SU(2)_L \times U(1)^2$. This is obtained when $R_4, \bar{R}_4$ components in the vector superfield remain zero modes $(e + w = 0)$. We consider the orbifold $T^2/\mathbb{Z}_4 \times T^2/\mathbb{Z}_3$ $(n_1 = 4, n_2 = 3)$. The charge assignments are

$$(x, y, z, w)_{Z_4} = (1, 3, 2, 2), \quad (x, y, z, w)_{Z_3} = (1, 1, 1, 0), \quad e_{Z_4} = 2, \quad e_{Z_3} = 0, \tag{32}$$

$$(k_1, l_1, m_1) = (1, 0, 3), \quad (k_2, l_2, m_2) = (1, 1, 1). \tag{33}$$

Then we obtain the zero modes in the chiral superfields as

$$
\begin{array}{c|c}
\Sigma_1 & (\mathbf{6}, \mathbf{2})_1, (\overline{\mathbf{15}}, \mathbf{1})_2 \\
\Sigma_2 & (\overline{\mathbf{15}}, \mathbf{1})_3 \\
\Sigma_3 & (\mathbf{6}, \mathbf{2})_2, (\overline{\mathbf{15}}, \mathbf{1})_1
\end{array}
\tag{34}
$$

The right-handed fermions are includes in the representation $(\overline{\mathbf{15}}, \mathbf{1})$, and the left-handed fermions are included in $(\mathbf{6}, \mathbf{2})$. So, there are two left-handed fermions and three right-handed fermions are the zero modes of the bulk fields in this example. One left-handed fermions must be a brane-localized field. The bulk interaction includes the Yukawa term

$$
(\mathbf{6}, \mathbf{2})_1(\mathbf{6}, \mathbf{2})_2(\overline{\mathbf{15}}, \mathbf{1})_3 + (\overline{\mathbf{15}}, \mathbf{1})_2(\overline{\mathbf{15}}, \mathbf{1})_3(\overline{\mathbf{15}}, \mathbf{1})_1.
\tag{35}
$$

The Higgs bidoublet components are also in the $(\mathbf{6}, \mathbf{2})_1$, and $(\mathbf{6}, \mathbf{2})_2$. In this example, the bulk Yukawa mass matrix for fermions is rank one, and only one of the eigenstate is massive from the bulk interaction.

4.2 Case 2

The three chiral superfields Σ_1, Σ_2, and Σ_3 include L_1, L_1, and L_2 components, respectively. To do this, it is necessary that

$$
- k_i + x = 0, \quad -l_i + x = 0, \quad -m_i + y = 0.
\tag{36}
$$

In this case, we can extract the zero modes of three chiral families, when (at least) two families are included in the adjoint representation. So, let us consider the example by using a model with E_7 bulk symmetry.

Example 3: We consider the orbifold $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$ ($n_1 = 6$, $n_2 = 3$). The charge assignments for the decomposed fields from E_7 are

$$
(x, y)_{Z_6} = (1, 4), \quad (x, y)_{Z_3} = (1, 1), \quad e_{Z_6} = 3, \quad e_{Z_3} = 0,
\tag{37}
$$

$$
(k_1, l_1, m_1) = (1, 1, 4), \quad (k_2, l_2, m_2) = (1, 1, 1).
\tag{38}
$$

Then, the 4D symmetry is $G_{\mathrm{PS}} \times U(1)^2$, and the zero modes are

$$
\begin{array}{c|c}
\Sigma_1 & L_1, \bar{R}_2', C' \\
\Sigma_2 & L_1', \bar{R}_2, C \\
\Sigma_3 & L_2, \bar{R}_1, H
\end{array}
\tag{39}
$$

Since the L_1 components in the superfields Σ_1 and Σ_2 are different fields, we denote them L_1 and L_1'. We obtain the massless modes for the three families and one Higgs bidoublet, and two sextet fields.

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The bulk interaction is

$$(L_1 \bar{R}_2 + L_1' \bar{R}_2')H + C(L_1 L_2 - \bar{R}_1 \bar{R}_2') + C'(L_1' L_2 + \bar{R}_1 \bar{R}_2). \tag{40}$$

One can find that (L_1, L_1'), $(\bar{R}_2, \bar{R}_2')$, and (C, C') form doublets under global $SU(2)$ symmetry, which originates from an R-symmetry for the gauge multiplet in the bulk. When the Higgs bidoublet H gets VEV, the fermions acquire masses. The two eigenvalues are degenerate and one eigenstate is massless.

4.3 Case 3

The three chiral superfields Σ_1, Σ_2, and Σ_3 include L_1, L_1, and L_1 components, respectively. To do this, it is necessary that

$$- k_i + x = 0, \quad -l_i + x = 0, \quad -m_i + x = 0. \tag{41}$$

Since $k_i = l_i = m_i$ needs to be satisfied, only $\mathbb{Z}_3$ orbifold is possible in this case.

The model can be constructed if (at least) one family is contained in the adjoint representation, and thus the bulk gauge symmetry can be $SU(8)$. We can construct a model by 6D $T^2/\mathbb{Z}_3$ orbifold with $SU(8)$ bulk gauge symmetry [20].

When we consider 8D $T^2/\mathbb{Z}_3 \times T^2/\mathbb{Z}_3'$ orbifold, one can obtain the model with $SO(16)$ and E_8 bulk gauge symmetries. Here, we give a solution for the case of bulk $SO(16)$ symmetry. The charge assignments for the decomposed fields from $SO(16)$, Eqs. (25,26), are

$$(a, b, c)_{Z_3} = (1, 2, 1), \quad (a, b, c)_{Z_3'} = (2, 2, 1), \tag{42}$$

$$(k_1, l_1, m_1) = (1, 1, 1), \quad (k_2, l_2, m_2) = (1, 1, 1). \tag{43}$$

In these solutions, there are three chiral families as well as three Higgs bidoublet fields as the zero modes from the bulk gauge multiplet. The bulk Yukawa coupling includes

$$\epsilon_{ijk} L_i \bar{R}_j H_k, \tag{44}$$

where ϵ_{ijk} is a totally anti-symmetric tensor. Note that i, j and k are not the indices from the $SU(4)$ subgroup in E_8, but they are the indices originate from Σ_1, Σ_2, and Σ_3. The Yukawa couplings for the zero modes have a global $SU(3)$ symmetry.

Since the Yukawa matrix is anti-symmetric, two eigenvalues of fermion masses are degenerate and one eigenstate is massless when the Higgs fields H_i get VEVs. The detail is given in the Ref.[20].

4.4 Case 4

Flavor $SU(2)$ gauge symmetry remains in 4D, and Σ_1 and Σ_2 include (L_1, L_2) and L_3 respectively. The field (L_1, L_2) in Σ_1 forms a doublet under the flavor $SU(2)$ symmetry. To satisfy this choice, it is necessary that

$$- k_i + x = 0, \quad x = y, \quad -l_i + z = 0. \tag{45}$$

Example 4: We consider the orbifold $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_4$ ($n_1 = 6$, $n_2 = 4$). The charge assignments for the decomposed fields are

$$(x, y, z, w)_{Z_6} = (1, 1, 4, 0), \quad (x, y, z, w)_{Z_4} = (1, 1, 1, 1), \quad e_{Z_6} = 3, \quad e_{Z_4} = 0, \tag{46}$$
$$(k_1, l_1, m_1) = (1, 4, 1), \quad (k_2, l_2, m_2) = (1, 1, 2). \tag{47}$$

Then, the 4D symmetry is $G_{\text{PS}} \times SU(2) \times U(1)^2$, and the zero modes are

$$\begin{array}{c|c}
\Sigma_1 & L_1, L_2, \bar{R}_3 \\
\Sigma_2 & L_3, \bar{R}_1, \bar{R}_2 \\
\Sigma_3 & C^{23}, C^{13}, H^{12}
\end{array} \tag{48}$$

The fields (L_1, L_2), $(\bar{R}_1, \bar{R}_2)$, and (C^{13}, C^{23}) form $SU(2)$ doublets. The bulk interaction is

$$(L_1 \bar{R}_2 + L_2 \bar{R}_1) H^{12} + L_3 (L_1 C^{13} + L_2 C^{23}) + \bar{R}_3 (-\bar{R}_1 C^{13} + \bar{R}_2 C^{23}). \tag{49}$$

When Higgs bidoublet H^{12} get a VEV, the fermions acquire masses. The two eigenvalues are degenerate, and one eigenstate (actually L_3 and $\bar{R}_3$) remain massless.

Example 5: We consider the orbifold $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_4$ ($n_1 = 6$, $n_2 = 4$). The charge assignments for the decomposed fields are

$$(x, y, z, w)_{Z_6} = (5, 5, 0, 2), \quad (x, y, z, w)_{Z_4} = (1, 1, 1, 1), \quad e_{Z_6} = 3, \quad e_{Z_4} = 0, \tag{50}$$
$$(k_1, l_1, m_1) = (1, 0, 5), \quad (k_2, l_2, m_2) = (2, 1, 1). \tag{51}$$

Then, the 4D symmetry is $G_{\text{PS}} \times SU(2) \times U(1)^2$, and the zero modes are

$$\begin{array}{c|c}
\Sigma_1 & C^{23}, C^{13}, H^{34} \\
\Sigma_2 & L_3 \\
\Sigma_3 & L_1, L_2, \bar{R}_4
\end{array} \tag{52}$$

In this example, three left-handed matters are zero modes, and one right-handed matter is zero modes. The fields (L_1, L_2) and (C^{13}, C^{23}) are $SU(2)$ doublets.

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The bulk interaction is

$$L_3 \bar{R}_4 H^{34} + L_3 (L_1 C^{13} + L_2 C^{23}). \tag{53}$$

When the Higgs bidoublet H^{34} get VEV, one eigenstate of fermion becomes massive. The other two left-handed fields (L_1, L_2) which is a flavor $SU(2)$ doublet, remain massless. They can become massive by brane localized terms. When we identify (L_1, L_2) as the first and second generations, one can construct a flavor $SU(2)$ model to obtain a hierarchical pattern.

4.5 Case 5

Flavor $SU(3)$ gauge symmetry remains in 4D, and Σ_1 includes (L_1, L_2, L_3). To satisfy this choice, it is necessary that

$$- k_i + x = 0, \quad x = y = z. \tag{54}$$

Example 6: We consider the the orbifold $T^2/\mathbb{Z}_4 \times T^2/\mathbb{Z}_3$ ($n_1 = 4$, $n_2 = 3$). The charge assignments for the decomposed fields are

$$(x, y, z, w)_{Z_4} = (1, 1, 1, 1), \quad (x, y, z, w)_{Z_3} = (1, 1, 1, 0), \quad e_{Z_4} = 2, \quad e_{Z_3} = 0, \tag{55}$$

$$(k_1, l_1, m_1) = (1, 0, 3), \quad (k_2, l_2, m_2) = (1, 1, 1). \tag{56}$$

Then, the 4D symmetry is $G_{\mathrm{PS}} \times SU(3) \times U(1)$, and the zero modes are

$$\begin{array}{c|c}
\Sigma_1 & L_1, L_2, L_3 \\
\Sigma_2 & H^{12}, H^{13}, H^{23}, S_1{}^4, S_2{}^4, S_3{}^4 \\
\Sigma_3 & \bar{R}_1, \bar{R}_2, \bar{R}_3
\end{array} \tag{57}$$

In this example, three families of both left- and right-handed matters are zero modes, and L_i, $\bar{R}_i$, $H^{ij}(= -H^{ji})$ are triplets under the flavor $SU(3)$ symmetry. The PS singlets $S_i{}^4$ also form a triplet under $SU(3)$.

The bulk interaction is

$$\sum_{i,j=1,2,3} L_i \bar{R}_j H^{ij}, \tag{58}$$

and the Yukawa matrix is anti-symmetric.

When the Higgs fields get VEVs, the flavor symmetry $SU(3)$ is broken down to $SU(2)$, and the two eigenvalues are degenerate, and one eigenstate is massless. (Surely, in a phenomenological model construction, the flavor symmetry must be broken at a higher scale by SM singlet VEVs.) The bulk Yukawa interaction is same as the case 3, where the $SU(3)$ symmetry in the bulk interaction is not gauged, while in this example, the flavor $SU(3)$ symmetry is a gauge symmetry.

Example 7: We consider the orbifold $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_4$ ($n_1 = 6$, $n_2 = 4$). The charge assignments for the decomposed fields are

$$(x, y, z, w)_{Z_6} = (2, 2, 2, 0), \quad (x, y, z, w)_{Z_4} = (1, 1, 1, 1), \quad e_{Z_6} = 3, \quad e_{Z_4} = 0, \tag{59}$$

$$(k_1, l_1, m_1) = (1, 2, 3), \quad (k_2, l_2, m_2) = (2, 1, 1). \tag{60}$$

Then, the 4D symmetry is $G_{\mathrm{PS}} \times SU(3) \times U(1)$, and the zero modes are

$$
\begin{array}{c|c}
\Sigma_1 & H^{14}, H^{24}, H^{34} \\
\Sigma_2 & L_1, L_2, L_3 \\
\Sigma_3 & \bar{R}_4
\end{array}
\tag{61}
$$

In this example, three families of left-handed matters and one right-handed matter are zero modes. The fields L_i, H^{i4} are fundamental and anti-fundamental representation under the flavor $SU(3)$ symmetry.

The bulk interaction is

$$\sum_{i=1,2,3} L_i \bar{R}_4 H^{i4}. \tag{62}$$

When the Higgs fields H^{i4} get VEVs, the flavor $SU(3)$ symmetry is broken down to $SU(2)$ symmetry. Only one of the eigenstate is massive, two other families can become massive by introducing the brane-localized interactions. In this situation, it is easy to construct a model with hierarchical fermion masses using the flavor symmetry [22].

5 Comments to build a model

As one can see, in the cases 1,2, and 4, the $SU(4)_c$ sextet fields are also the zero modes from the bulk fields. In the case 1, one can find that the $SU(4)_c$ sextet components are embedded in **27** in Example 1 and $(\overline{\mathbf{15}}, \mathbf{1})$ in Example 2. The reason for this is as follows: In these cases, we need to satisfy

$$- k_i + x = 0, \quad -l_i + y = 0, \tag{63}$$

up to the permutation of (k_i, l_i, m_i) and (x, y, z, w). Because of a relation $k_i + l_i + m_i = 0$, we need $x + y + m_i = 0$. Since $-x - y$ is a charge of a $SU(4)$ sextet, the corresponding sextet must be a zero mode.

If one thinks that the sextet field is unwanted, one needs to choose case 3 or 5. Actually, the sextet fields contain colored Higgs fields which may generate a dangerous dimension 5 operator for proton decay. So, if we add a brane-localized term, a dangerous proton decay operator may be introduced. However, the sextet fields have extra $U(1)$ charges, and one can forbid the

dangerous terms if we do not introduce random PS singlet fields which acquire VEV to break the extra $U(1)$ symmetries. Furthermore, since the representation of the (anti-fundamental) colored-Higgs fields is same as the right-handed down-type quarks, one can interpret that the $SU(4)_c$ sextet contains the right-handed quarks, and it can be available to break the strange quark and muon mass unification (to obtain so-called Georgi-Jarskog relation). It can be realized by mixing the fields with the right-handed quark in $\bar{R} : (\bar{4}, 1, 2)$. The mixing terms will be introduced as a brane-localized term. Or, one of $\bar{R}$ can be identified a Higgs field to break Pati-Salam symmetry broken down to SM gauge symmetry. In that case, the bulk term $C\bar{R}\bar{R}'$ make $(3, 1)_{-1/3}$ component in C massive, and the down-type quark component in C remain massless.

To construct a model, we need to care about the brane-localized gauge anomalies [23]. In general, the set of 4D zero modes from the bulk chiral superfields causes the gauge anomalies, and one has to introduce brane-localized fields at each 4D fixed point to cancel the anomalies. To cancel them, one has to introduce the fields not to increase (or decrease) the number of chiral families since our aim is to obtain the three chiral families from the bulk. The introduction of the brane fields may be complicated, but since it is not an essential part, we do not focus it in this paper.

The main focus in this paper is the Yukawa coupling structure originating from the bulk gauge interaction. The Yukawa matrix originating from the bulk interaction is classified for the following three situations as we have seen in the Examples in the previous section:

(a) All three families can be massive.

(b) The two eigenvalues are degenerate, and one of the eigenstate is massless.

(c) One of the families is massive, and other two are massless.

The situation (a) is a special case among the five cases in the previous section. This is obtained from the case 1 in which the Yukawa interaction does not have any global or gauged non-Abelian flavor symmetry. In this case, however, if we see the Pati-Salam branch, the 4D gauge symmetry is always larger than the PS symmetry, and to obtain the situation (a), E_6 symmetry remains in the 4D fixed points. In that case, the Yukawa matrix is symmetric under the flavor indices and the diagonal elements are all zero. As a consequence, the Yukawa matrix can not have three hierarchical eigenvalues (namely, $m_1 \ll m_2 \ll m_3$). To obtain realistic hierarchical fermion masses, we surely need an assist from the brane-localized terms.

In the situation (b), the Yukawa matrix is an anti-symmetric mass matrix due to global or local $SU(3)$ symmetry, or there is a global or local $SU(2)$ flavor symmetry in the bulk interaction. This case is interesting to explain why the first generation of fermions has tiny mass as is pointed out in Ref.[20]. To resolve the degeneracy between second and third generation,

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we need to introduce the brane-localized term, which is also studied in the Ref.[20].

In the situation (c), only one family becomes massive because three left-handed matters are in the bulk, but only one of the right-handed matters is in the bulk as in the Examples 5 and 7. This situation provides a good example to explain why only third generation is heavy and their Yukawa coupling constant can be unified to the gauge coupling. It is interesting that the flavor symmetry remains in 4D in the Examples to construct a flavor model which can explain the hierarchy between first and second generation.

Among the three situations above, the situation (a) is special, and it is possible to have a different fermion mass hierarchy, if we see another branch of bulk gauge symmetry breaking. Another typical branch which include SM gauge symmetry is a trinification symmetry $G_{333} = SU(3)_c \times SU(3)_L \times SU(3)_R$. Since E_6 has a maximal subgroup G_{333}, it is possible to break the E_8 symmetry down to $E_6 \times SU(3) \to G_{333} \times U(1)^2$ by orbifold [17].

6 Trinification branch and SM decomposition

6.1 Charge assignments for trinification

The E_8 group has a E_6 branch as in Eq.(30), and E_6 has a trinification subgroup. The adjoint **78** and fundamental representation **27** are decomposed under G_{333} as follows:

$$\mathbf{78} = (\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8}) + (\mathbf{3},\mathbf{3},\mathbf{3}) + (\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}}), \tag{64}$$

$$\mathbf{27} = (\mathbf{3},\bar{\mathbf{3}},\mathbf{1}) + (\mathbf{1},\mathbf{3},\bar{\mathbf{3}}) + (\bar{\mathbf{3}},\mathbf{1},\mathbf{3}). \tag{65}$$

The representation **27** includes $Q : (\mathbf{3},\bar{\mathbf{3}},\mathbf{1})$, $Q^c : (\mathbf{1},\mathbf{3},\bar{\mathbf{3}})$, and $H : (\bar{\mathbf{3}},\mathbf{1},\mathbf{3})$, which include all the fermion species as well as Higgs representations under SM decomposition. Note that the decomposed representation **27** in E_8 has a $SU(3)$ flavor index.

The discrete charges are assigned to the decomposed representations as follows: For the adjoint representations for E_6 and $SU(3)$, the charges are

$$
\begin{array}{|c|c|c||c|}
\hline
(\mathbf{8},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{8}) & (\mathbf{3},\mathbf{3},\mathbf{3}) & (\bar{\mathbf{3}},\bar{\mathbf{3}},\bar{\mathbf{3}}) & S_i{}^j \\
\hline
0 & a & -a & x_i - x_j \\
\hline
\end{array}
\tag{66}
$$

where $S_i{}^j$ is a $SU(3)$ adjoint representation.

The matter and anti-matter representations are given as

$$
\begin{array}{|c|c|c||c|c|c|}
\hline
Q_i & Q_i^c & H_i & \bar{Q}^i & \bar{Q}^{ci} & \bar{H}^i \\
\hline
x_i & a + x_i & -a + x_i & -x_i & -a - x_i & a - x_i \\
\hline
\end{array}
\tag{67}
$$

We need conditions $\sum x_i \equiv 0$, $3a \equiv 0 \pmod{n}$.

Example 8: Let us obtain the case 1 solution considered in section 4 in $T^2/\mathbb{Z}_3 \times T^2/\mathbb{Z}_3'$ orbifold. The discrete charges are assigned as

$$(x_1, x_2, x_3)_{Z_3} = (1,1,1), \quad (x_1, x_2, x_3)_{Z_3'} = (0,1,2), \quad a_{Z_3} = 0, \quad a_{Z_3'} = 1, \tag{68}$$

$$(k_1, l_1, m_1) = (1,1,1), \quad (k_2, l_2, m_2) = (0,1,2). \tag{69}$$

Then, the 4D symmetry is $G_{333} \times U(1)^2$, and the zero modes are

$$
\begin{array}{c|l}
\Sigma_1 & Q_1, Q_3^c, H_2 \\
\Sigma_2 & Q_2, Q_1^c, H_3 \\
\Sigma_3 & Q_3, Q_2^c, H_1
\end{array}
\tag{70}
$$

The bulk interaction includes

$$Q_1 Q_2^c H_3 + Q_2 Q_3^c H_1 + Q_3 Q_1^c H_2 + Q_1 Q_2 Q_3 + Q_1^c Q_2^c Q_3^c + H_1 H_2 H_3. \tag{71}$$

Quarks q, u^c, d^c, leptons ℓ, e^c, ν^c and the Higgs fields are embedded in Q, Q^c and H as

$$Q = (q, h^C), \qquad Q^c = (u^c, d^c, \bar{h}^C), \qquad H = \begin{pmatrix} h^u & h^d & L \\ e^c & \nu^c & s \end{pmatrix}, \tag{72}$$

where h^u, h^d are Higgs doublets, h^C and $\bar{h}^C$ are colored Higgs component, and s is a singlet component under SM. Since they are embedded in **27** representation of E_6, the prediction of weak mixing angle is kept to be $\sin^2 \theta_W = 3/8$ (as long as brane-localized contribution is suppressed). Then, the bulk interaction includes the terms in terms of the SM decomposed fields as

$$Q_1 Q_2^c H_3 = q_1 u_2^c h_3^u + q_1 d_2^c h_3^d + q_1 \bar{h}_2^C \ell_3 + h_1^C d_2^c e_3^c + h_1^C u_2^c \nu_3^c + h_1^C \bar{h}_2^C s_3, \tag{73}$$

$$Q_1 Q_2 Q_3 = q_1 q_2 h_3^C + q_2 q_3 h_1^C + q_3 q_1 h_2^C, \tag{74}$$

$$Q_1^c Q_2^c Q_3^c = u_1^c d_2^c \bar{h}_3^C + u_2^c d_3^c \bar{h}_1^C + u_3^c d_1^c h_2^C + d_1^c u_2^c \bar{h}_3^C + d_2^c u_3^c \bar{h}_1^C + d_3^c u_1^c h_2^C, \tag{75}$$

$$H_1 H_2 H_3 = \ell_1 e_2^c h_3^d + \ell_1 \nu_2^c h_3^u + s_1 h_2^u h_3^d + (\text{permutation of indices}). \tag{76}$$

Because Q_1^c component in Σ_1 is projected out for example, the Yukawa matrix for quarks is no more symmetric. Therefore, we do not have a restriction which we found in Example 1. In a proper basis, Yukawa couplings for all three generations of quarks are unified to the gauge coupling. When the ratios of VEVs of Higgs doublet fields $h_i^{u,d}$ are all free, one can have any ratios of three-generation quark masses. On the other hand, the lepton Yukawa matrix is still symmetric because H_i multiplet includes both left- and right-handed leptons. As it is mentioned, the symmetric matrix without diagonal elements can not have three hierarchical eigenvalues.

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It is preferable that only one of the linear combination of Higgs doublet fields is light to construct a low energy model in order to avoid too large flavor changing neutral currents. The gauge coupling unification also prefers the situation. The bulk fields H_i contain the standard model singlets s_i as well as $SU(2)_L$ doublets h_i^u, h_i^d, and the bulk interaction $H_1 H_2 H_3$ includes the doublet Higgs mass terms $m_{ij} h_i^u h_j^d$ when s_i's acquire VEVs,

$$
m_{ij} = \begin{pmatrix} 0 & s_3 & s_2 \\ s_3 & 0 & s_1 \\ s_2 & s_1 & 0 \end{pmatrix}. \tag{77}
$$

The mass matrix m_{ij} is symmetric and there is no diagonal elements since it comes from the bulk interaction and H_i multiplet includes both h_i^u and h_i^d. When the VEVs of singlets are hierarchical ($s_3 \ll s_2 \ll s_1$, for example), only one of the linear combination of Higgs double fields is light, and the Higgs mixings are hierarchical. Then, the VEVs of Higgs fields will be hierarchical, $\langle h_3^{u,d} \rangle \ll \langle h_2^{u,d} \rangle \ll \langle h_1^{u,d} \rangle$, and the quark masses are hierarchical. Or, one can interpret that the effective Yukawa couplings for quarks are (unified) gauge couplings multiplied with the hierarchical Higgs mixings. In this choice, one of mass eigenvalues of lepton is hierarchically small, but two eigenvalues are degenerate due to the symmetricity of the matrix.

To obtain the VEVs of singlets s_i, one needs brane-localized terms. The hierarchy of the VEVs may be related to the volume of each torus, since the brane-localized terms needs to be generated by Wilson line operators.

In order to obtain a phenomenological model, we have to break up and down symmetry for quark masses, and flavor mixings have to be introduced. It can be realized when brane-localized terms are introduced as usual construction of a trinification model. Also we need to break the degeneracy of charged-lepton masses. It can be done by introducing vector-like brane-localized fields which is mixed with the bulk fields, and the second generation of charged-lepton is replaced to the brane-localized field.

We note that the bulk interaction also contains colored Higgs couplings, e.g. qqh^C, $q\ell\bar{h}^C$ etc. They can generate the dimension five operator for proton decay. Actually, colored Higgs masses are also generated when the singlet components s_i acquire VEVs. When the VEVs are hierarchical, there are light colored Higgs fields, and it causes a rapid proton decay. Even if we add mass terms of colored Higgs fields, the colored Higgs couplings are still dangerous when the fermion mass hierarchy comes from doublet Higgs mixings keeping the original coupling is unified to the gauge coupling as noted above. Since the colored Higgs fields do not have hierarchical mixing, the colored Higgs couplings does not have flavor suppression and it generate too large nucleon decay amplitudes. To avoid it, the bulk fields should be replaced with a brane field by introducing vector-like brane matter. Or, the colored Higgs fields can be projected out

by adopting additional orbifold projection. In the latter case, however, some of the generations are also projected out. For example, suppose that all h_i^C are projected out. When the discrete charge of h_3^C is not zero, the discrete charge of one of q_1 and q_2 component can not be zero due to the conservation of the discrete charge in the coupling $q_i q_j h_k^C$. As a result, two of the quark doublets q_i are projected out. Then, the quark mass matrix from bulk interaction has to be (at most) rank 1. Similarly, due to the coupling $u^c e^c h^C$, at least one of the right-handed charged-lepton is projected out. Therefore, if the colored Higgs fields are projected out by orbifold in this branch, we obtain as a consequence the first generation of quarks and leptons to be always massless when the brane-localized terms are suppressed.

We also note that E_7 adjoint has one flavor of trinification matter. Therefore, we obtain the three family trinification model in E_7 bulk symmetry using the case 3 solution. In that case, the Yukawa matrix is anti-symmetric, and thus the two eigenvalues are degenerate, and one eigenvalue is zero.

6.2 SM decomposition

As we have noted, under the Pati-Salam branch, $SU(4)_c$ sextet $(\mathbf{6}, \mathbf{1}, \mathbf{1})$ can include the right-handed down-type quark. Similarly, when the PS symmetry is broken down to SM gauge symmetry, $(\mathbf{6}, \mathbf{2}, \mathbf{2})$, $(\mathbf{15}, \mathbf{1}, \mathbf{1})$, $(\mathbf{1}, \mathbf{2}, \mathbf{2})$, and $(\mathbf{1}, \mathbf{1}, \mathbf{3})$ representations contain the quark doublet, right-handed up-type quark, lepton doublet, and right-handed charged lepton, respectively. Therefore, one more family can be included in the adjoint representation. In E_7 and E_8, the number of family is three and five, respectively, which can be found if we see the branches, $E_7 \rightarrow SU(8)$ (or $SU(6) \times SU(3)$) $\rightarrow SU(5) \times SU(3) \times U(1)$ and $E_8 \rightarrow SU(5) \times SU(5)$:

$$\mathbf{133} = (\mathbf{24}, \mathbf{1}) + (\mathbf{10}, \overline{\mathbf{3}}) + (\overline{\mathbf{10}}, \mathbf{3}) + (\mathbf{5}, \overline{\mathbf{3}}) + (\overline{\mathbf{5}}, \mathbf{3}) + (\mathbf{5}, \mathbf{1}) + (\overline{\mathbf{5}}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}) + (\mathbf{1}, \mathbf{1}), \tag{78}$$

$$\mathbf{248} = (\mathbf{24}, \mathbf{1}) + (\mathbf{10}, \overline{\mathbf{5}}) + (\overline{\mathbf{10}}, \mathbf{5}) + (\mathbf{5}, \mathbf{10}) + (\overline{\mathbf{5}}, \overline{\mathbf{10}}) + (\mathbf{1}, \mathbf{24}). \tag{79}$$

The discrete charge assignments for the SM decomposed representations in the adjoint of E_7 and E_8 are given in Appendix.

One can find that if we assign all of three quark doublets to have zero modes for the case 1 solution ($x_1 + x_2 + x_3 = 0$, for example), $(\mathbf{1}, \mathbf{2})_{\pm 1/2}$ components remain massless in the vector multiplet for both E_7 and E_8 cases. Therefore, in the case 1 solution, $SU(3)_L (\supset SU(2)_L)$ symmetry always remains in 4D. As a consequence, when the gauge symmetry is broken down to SM and the global or gauged flavor symmetry is completely broken to $U(1)$'s, (at least) one of generation of quark is massless in the limit where the brane-localized couplings are zero. In the case where the bulk gauge symmetry is E_8 and the trinification G_{333} or E_6 symmetry remains in 4D, all three generation can become massive as we have seen. (In this case, at least $SU(3)_L$

symmetry remains in 4D.) This is because $x_1 + x_2 + x_3 = 0$ is satisfied (up to permutation of $SU(5)$ flavor index) for these two branches. However, as we have noted, the dangerous dimension five operators can be generated. Therefore, one of generation of quark fields should be replaced to a brane-localized field. Or, the colored Higgs fields should be projected out by additional orbifold condition. In any cases, the mass of first generation should be small when the flavor symmetries are completely broken and the brane-localized couplings are suppressed.

Here, we consider the case where the 4D symmetry is the standard model, and three generations of right-handed quarks u^c are obtained from bulk.

Example 9: Let us obtain the case 1 solution considered in section 4 in $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$ orbifold in E_7 bulk symmetry. The discrete charges are assigned as

$$(x_1, x_2, x_3)_{Z_6} = (0, 1, 2), \quad (x_1, x_2, x_3)_{Z_3} = (1, 1, 1), \quad a_{Z_6} = 1, \quad a_{Z_3} = 0, \tag{80}$$

$$(k_1, l_1, m_1) = (1, 2, 3), \quad (k_2, l_2, m_2) = (1, 1, 1). \tag{81}$$

Then, the 4D symmetry is $G_{\mathrm{SM}} \times U(1)^3$, and the zero modes are

$$\begin{array}{c|c} \Sigma_1 & q_2, u_1^c, e_3^c \\ \Sigma_2 & q_3, u_2^c, \bar{\ell}^{23} \\ \Sigma_3 & u_3^c, \bar{\ell}^{13}, \bar{d}^{c23} \end{array} \tag{82}$$

The three right-handed up-type quarks are the zero modes of bulk fields. The bulk interaction includes

$$q_2 u_3^c \bar{\ell}^{23} + q_3 u_1^c \bar{\ell}^{13} + u_2^c e_3^c \bar{d}^{c23}. \tag{83}$$

The fields $\bar{\ell}^{23}$ and $\bar{\ell}^{13}$ can be considered as up-type Higgs fields. When the Higgs fields acquire VEVs, two eigenvalues of up-type quarks become non-zero. The eigenvalues can be hierarchical when the Higgs mixing is small due to the suppression of couplings in the brane-localized interaction. In this example, one can construct the model in 8D, and thus, the scalar component Σ_3 can be chosen not to be gauge fields, but, Σ_1 and Σ_2 can be considered the gauge fields with extra dimensional components. Then, the light linear combination of the Higgs fields can be almost $\bar{\ell}^{23}$ rather than $\bar{\ell}^{13}$, since the coupling with $\bar{\ell}^{23}$ will be generated by a Wilson line operator and it is expected to be small [20].

In the bulk E_8 case, we can find a similar solution as Example above. However, it is rare to obtain both up- and down-type quarks as well as charged leptons to get masses if the symmetry is broken down to SM, because many of the components are projected out in that case. If some of the symmetry remains like $SU(4)_c \times SU(2)_L \times U(1)_R$ symmetry or $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ symmetry, one can obtain their Yukawa couplings (namely, top, bottom, and tau Yukawa couplings) from the bulk gauge interaction. As we have

noted, in such 4D symmetries, all three generations of left-handed quarks cannot have zero modes when bulk flavor symmetry is completely broken.

As we have seen in section 4, in the cases 2 and 4 where $SU(2)$ flavor symmetry remains in 4D, two of the mass eigenvalues are degenerate in the Pati-Salam branch (namely, the mass matrix is rank 1 or eigenvalues are degenerate even if the rank is 2). This is because of the restriction of discrete charge assignment $2e = 0$ in the Pati-Salam branch. When the 4D symmetry is G_{SM}, we have additional examples where the mass matrix is rank 2 and the eigenvalues are not degenerate. Here we give an example in the case where gauged $SU(2)$ flavor symmetry remains in 4D.

Example 10: Let us obtain the case 4 solution considered in section 4 in $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3$ orbifold in E_8 bulk symmetry. The discrete charges are assigned as

$$(x_1, x_2, x_3)_{Z_6} = (1, 1, 2, 3, 4), \quad (x_1, x_2, x_3)_{Z_3} = (1, 1, 1, 0, 0), \quad a_{Z_6} = 1, \quad a_{Z_3} = 0, \quad (84)$$

$$(k_1, l_1, m_1) = (1, 2, 3), \quad (k_2, l_2, m_2) = (1, 1, 1). \quad (85)$$

Then, the 4D symmetry is $G_{SM} \times SU(2) \times U(1)^2$, and the zero modes are

$$
\begin{array}{c|c}
\Sigma_1 & q_1, q_2, e_3^c, \ell_{35} \\
\Sigma_2 & q_3, u_1^c, u_2^c, \bar{\ell}^{13}, \bar{\ell}^{23} \\
\Sigma_3 & u_3^c, \bar{\ell}^{12}, \bar{d}^{c13}, \bar{d}^{c23}, S_1^5, S_2^5
\end{array}
\qquad (86)
$$

The bulk interaction includes

$$(q_1 u_2^c + q_2 u_1^c)\bar{\ell}^{12} + (q_1 \bar{\ell}^{13} + q_2 \bar{\ell}^{23})u_3^c + (u_1 \bar{d}^{c13} + u_2 \bar{d}^{c23})e_3^c + \ell_{35}(\bar{\ell}^{13} S_1^5 + \bar{\ell}^{23} S_2^5). \qquad (87)$$

The fields $\bar{d}^{c13}, \bar{d}^{c23}$ are colored Higgs, and $\bar{\ell}^{12,13,23}$ are up-type Higgs fields. The field ℓ_{35} can be considered as a lepton doublet, and then, S_1^5 and S_2^5 corresponds to right-handed neutrinos. Note that $q\ell\bar{d}^c$ coupling is forbidden. The field ℓ_{35} can be also considered as a Higgs doublet, and S_1^5 and S_2^5 form a $SU(2)$ flavor doublet. When S_1^5 acquires a VEV, the flavor $SU(2)$ symmetry is broken, and ℓ_{35} and $\bar{\ell}^{13}$ become massive. Then, the quark mass matrix from the bulk interaction is

$$
(q_3, q_1, q_2)
\begin{pmatrix}
0 & 0 & 0 \\
0 & \langle\bar{\ell}^{12}\rangle & 0 \\
\langle\bar{\ell}^{12}\rangle & 0 & \langle\bar{\ell}^{23}\rangle
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}. \qquad (88)
$$

As is the previous example, the Higgs mixing can become small, and the two VEVs are hierarchical. Then, two eigenvalues are hierarchical, and one eigenvalue is zero.

One can also consider a similar solution where global $SU(2)$ flavor symmetry remains. Also, in the case of E_8 bulk symmetry, the discrete charges can be assigned so that both global and gauged flavor symmetries remain in 4D.

22

7 Conclusion

We studied the higher dimensional models in which gauge, Higgs and three families of matter fields are unified in a SUSY gauge multiplet. The gauge multiplet contains three chiral superfields Σ_i as well as a vector multiplet in 4D point of view. When both Higgs and matter fields are extracted as zero modes by orbifold projection, the Yukawa interaction in 4D is generated from the bulk gauge interaction.

We classified the cases in which three generations are included in the zero modes of the bulk superfields. The bulk gauge symmetry is broken down to the 4D gauge symmetry which contains the SM gauge group by orbifolding, and some of non-Abelian gauged flavor symmetry can also remain in 4D. For example, E_8 has a subgroup $SO(10) \times SU(4)$, and thus in the $SO(10)$ basis, there can be maximally $SU(4)$ gauged flavor symmetry. Furthermore, since there are three chiral superfields, there is a global symmetry which originates from R symmetry in the bulk. If the gravity is taken into account, the R symmetry is also gauged, but we consider it as a global symmetry in a flat limit. Therefore, to obtain three generations, there are cases where the gauged or global flavor symmetry remains. One can assign to make $SU(2)$ and $SU(3)$ flavor symmetries remain for both global and gauged symmetries. Also, it can be considered that the both global and gauged flavor symmetry is completely broken by orbifolding. Totally, there are 5 cases to obtain three generations from the bulk fields. We investigated each case and gave examples for the discrete charge assignment of the orbifold.

Due to the bulk flavor symmetry, the Yukawa structure originated from the bulk gauge interaction is restricted. Many of the cases, two eigenvalues are degenerate in the limit where the couplings in brane-localized terms are zero. There are two situations in those cases: Two eigenvalues are degenerate and one eigenvalue is zero. Or, two eigenvalues are zero, and one eigenstate is massive. Since the brane-localized terms are suppressed by a factor from the volume of the extra dimension and the bulk interaction gives a dominant contribution to the Yukawa couplings, the former situation is not a good situation phenomenologically. To make a phenomenologically viable model, one needs to mix one of the eigenstate with a brane field, and the brane field is a light eigenstate of our quarks and leptons. In the latter situation, on the other hand, one can explain why only third generation is heavy. The two other generations can become massive by brane terms. The hierarchy between first and second generation can be explained by the remaining flavor symmetry.

When the non-Abelian flavor symmetry for both global and gauged symmetry is broken by orbifold, the degeneracy of eigenvalues can be resolved. In that case, the discrete charge assignment is restricted, and higher gauge symmetry often remains in 4D. Actually, when all three quark doublets have zero modes in that case, at least $SU(3)_L$ gauge symmetry (which contains

$SU(2)_L$) remains in 4D. In other words, if the gauge symmetry is broken down to $SU(2)_L$, all three quark doublets can not have zero modes, and at least one of the eigenvalues is zero. It can interestingly explain why the first generation mass is tiny. We gave an example where all three generations of quarks have non-degenerate masses from the bulk Yukawa interaction in the model in which trinification symmetry remains in 4D. We also gave an example where two of the up-type quarks have non-degenerate eigenvalues from the bulk Yukawa interaction with 4D standard model gauge group.

In conclusion, the Yukawa coupling is generated from the bulk gauge interaction when left- and right-handed matter fields as well as Higgs fields are zero modes of the bulk fields. We considered the cases where three generations of matter are contained in the zero modes. In that cases, due to the bulk flavor symmetries, the structure of bulk Yukawa coupling is restricted and the hierarchy of fermion masses can be explained by a nature of extra dimensions.

Appendix : Charge assignments for SM decomposed representations

We obtain the discrete charge assignments for the SM decomposed representations in the E_7 adjoint. The branch $E_7 \to SU(5) \times SU(3) \times U(1)$ is useful to arrange the representations.

For the adjoint representations for G_{SM} and $SU(3)$, the charges are

$$
\begin{array}{|c|c|c||c|}
\hline
(\mathbf{8},\mathbf{1})_0 + (\mathbf{1},\mathbf{3})_0 + (\mathbf{1},\mathbf{1})_0 & (\mathbf{3},\mathbf{2})_{-5/6} & (\bar{\mathbf{3}},\mathbf{2})_{5/6} & S_i{}^j \\
\hline
0 & a & -a & x_i - x_j \\
\hline
\end{array}
\tag{89}
$$

where $S_i{}^j$ is a $SU(3)$ adjoint representation ($i = 1, 2, 3$).

The matter and anti-matter representations are given as follows:

Discrete charges for the representations in $(\mathbf{10}, \bar{\mathbf{3}})$ and $(\overline{\mathbf{10}}, \mathbf{3})$ are

$$
\begin{array}{|c|c|c||c|c|c|}
\hline
q_i & u_i^c & e_i^c & \bar{q}^i & \bar{u}^{ci} & \bar{e}^{ci} \\
\hline
x_i & a + x_i & -a + x_i & -x_i & -a - x_i & a - x_i \\
\hline
\end{array}
\tag{90}
$$

The representations in $(\bar{\mathbf{5}}, \mathbf{3})$ and $(\mathbf{5}, \bar{\mathbf{3}})$ are

$$
\begin{array}{|c|c||c|c|}
\hline
d_{ij}^c & \ell_{ij} & \bar{d}^{cij} & \bar{\ell}^{ij} \\
\hline
x_i + x_j & x_i + x_j + a & -x_i - x_j & -x_i - x_j - a \\
\hline
\end{array}
\tag{91}
$$

The representations in $(\bar{\mathbf{5}}, \mathbf{1})$ and $(\mathbf{5}, \mathbf{1})$ are

$$
\begin{array}{|c|c||c|c|}
\hline
D^c & \bar{H} & \bar{D}^c & H \\
\hline
-x_1 - x_2 - x_3 - a & -x_1 - x_2 - x_3 & x_1 + x_2 + x_3 + a & x_1 + x_2 + x_3 \\
\hline
\end{array}
\tag{92}
$$

When $x_3 + a \equiv 0$ is satisfied for example, $SU(4)_c \times SU(2)_L \times U(1)_R$ symmetry remains in 4D. When $x_3 - a \equiv 0$ is satisfied, $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ symmetry remains. When $x_3 + a \equiv 0$ and $2a \equiv 0$ are satisfied, Pati-Salam symmetry remains. When $x_2 \equiv x_3 \equiv a$ and $3a \equiv 0$ are satisfied, trinification symmetry remains. When $a \equiv 0$ is satisfied, $SU(5)$ symmetry remains. When $x_3 \equiv 0$ is satisfied, flipped-$SU(5)$ remains. When $x_3 \equiv a \equiv 0$ is satisfied, $SO(10)$ symmetry remains. When $x_2 \equiv x_3 \equiv a \equiv 0$ is satisfied, E_6 symmetry remains.

Next, let us obtain the discrete charge assignments for the SM decomposed representations in the E_8 adjoint. The branch $E_8 \to SU(5) \times SU(5)$ is useful to arrange the representations.

For the adjoint representations for G_{SM} and $SU(5)$, the charges are

$$
\begin{array}{|c|c|c||c|}
\hline
(\mathbf{8}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{1})_0 & (\mathbf{3}, \mathbf{2})_{-5/6} & (\bar{\mathbf{3}}, \mathbf{2})_{5/6} & S_i{}^j \\
\hline
0 & a & -a & x_i - x_j \\
\hline
\end{array}
\tag{93}
$$

where $S_i{}^j$ is a $SU(5)$ adjoint representation $(i = 1, 2, 3, 4, 5)$.

The matter and anti-matter representations are given as follows:

Discrete charges for the representations in $(\mathbf{10}, \bar{\mathbf{5}})$ and $(\overline{\mathbf{10}}, \mathbf{5})$ are

$$
\begin{array}{|c|c|c||c|c|c|}
\hline
q_i & u_i^c & e_i^c & \bar{q}^i & \bar{u}^{ci} & \bar{e}^{ci} \\
\hline
x_i & a + x_i & -a + x_i & -x_i & -a - x_i & a - x_i \\
\hline
\end{array}
\tag{94}
$$

The representations in $(\bar{\mathbf{5}}, \overline{\mathbf{10}})$ and $(\mathbf{5}, \mathbf{10})$ are

$$
\begin{array}{|c|c||c|c|}
\hline
d_{ij}^c & \ell_{ij} & \bar{d}^{cij} & \bar{\ell}^{ij} \\
\hline
x_i + x_j & x_i + x_j + a & -x_i - x_j & -x_i - x_j - a \\
\hline
\end{array}
\tag{95}
$$

From the algebra, the condition $\sum x_i + a \equiv 0$ has to be satisfied.

Similarly to the E_7 case, higher symmetries remain in 4D when above conditions are satisfied.

Acknowledgments

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